

MSC2010 17D99

# ABOUT SOME VARIETIES OF LEIBNITZ ALGEBRAS

Yu. Yu. Frolova, T. V. Skoraya

The paper presents two new results concerning the varieties of Leibnitz algebras. In the case of prime characteristic  $p$  of the base field constructed example not nilpotent variety of Leibnitz algebras satisfying an Engel condition order  $p$ . In the case of zero characteristic obtained new results concerning the space of multilinear elements variety of left nilpotent of the class not more than 3 Leibnitz algebras.

## 1. INTRODUCTION

Recall that a Leibnitz algebra is a vector space over a field with bilinear product, which satisfies to the identity:

$$(xy)z \equiv (xz)y + x(yz). \quad (1)$$

According to this identity multiplication on an element of algebra becomes differentiation of this algebra. On condition of performance identity of anti-commutativity  $xy \equiv -yx$ , Leibnitz identity is equivalent to the Jacobi identity:  $x(yz) + y(zx) + z(xy) \equiv 0$ . Therefore if The Leibnitz algebra satisfies to the identity  $xx \equiv 0$ , then it is a Lie algebra. In particular, any Lie algebra is a Leibnitz algebra. The converse is not true. . Note that probably first the class of Leibnitz algebras was considered in paper [2] as a generalization of the notion of Lie algebras.

We will transform Leibnitz identity as follows:  $x(yz) \equiv (xy)z - (xz)y$ . According to the received identity any element of Leibnitz algebra can be presented in linear combination of elements in which brackets are arranged from left to right. Therefore we will lower brackets in case of their left arrangement, i.e.  $((x_1x_2)x_3) \dots x_n = x_1x_2x_3 \dots x_n$ . For convenience we denote the operator of multiplication on the right, for example, on an element  $z$  a capital letter of  $Z$ , assuming that  $xz = xZ$ . In particular, in our notation we obtain  $\underbrace{xyy \dots y}_m = xY^m$ .

A set of algebras, that satisfy some fixed set of identities, is called a variety of linear algebras.

Let the base field  $\Phi$  has zero characteristic. In this case all information on variety contains in multilinear elements of its algebras.

Let  $F(X, \mathbf{V})$  be a relatively free algebra of variety  $\mathbf{V}kj$  with a countable set of free generators  $X = \{x_1, x_2, \dots\}$  and let  $P_n = P_n(\mathbf{V})$  be a set of all multilinear elements from  $x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)}$  in  $F(X, \mathbf{V})$ . Note that in the future, for the sake of presentation, we will denote generators of the relatively

free algebra as well other symbols. Let  $\sigma$  be an element of symmetric group  $S_n$ . We assume that as a result of the actions of the left permutation  $\sigma$  on element  $x_{i_1}x_{i_2}\dots x_{i_n}$  space  $P_n$  we get an element  $x_{\sigma(i_1)}x_{\sigma(i_2)}\dots x_{\sigma(i_n)}$ . This sets the action of  $S_n$  on the space  $P_n$ , in consequence of which  $P_n$  becomes a module over the group ring of  $\Phi S_n$ . Structure of  $P_n$  as  $\Phi S_n$ -module plays an important role and is actively studied for different varieties.

Recall that the standard polynomial of degree  $n$  has the form:

$$St_n(x_1, x_2, \dots, x_n) = \sum_{q \in S_n} (-1)^q x_{q(1)} x_{q(2)} \dots x_{q(n)},$$

where summation is over the elements of symmetric group, and  $(-1)^q$  is equal to  $+1$  or  $-1$  depending on the parity of the permutation  $q$ . Agree variables in standard polynomial denoted by special symbols on top (heck, wave and so on). For example, the standard polynomial of degree  $n$  from the variables  $x_1, x_2, \dots, x_n$  will be written as follows:  $St_n = \bar{x}_1, \bar{x}_2, \dots, \bar{x}_n$ . As a sign of this polynomial depends on parity permutations, its variables will call skew-symmetric. Variables in different skew-symmetric sets will be denoted by different symbols, for example:

$$\sum_{q \in S_n, p \in S_m} (-1)^{q+p} x_{q(1)} x_{q(2)} \dots x_{q(n)} y_{p(1)} y_{p(2)} \dots y_{p(m)} = \bar{x}_1 \bar{x}_2 \dots \bar{x}_n \tilde{y}_1 \tilde{y}_2 \dots \tilde{y}_m.$$

Note that if the element contains the same variables in different skew-symmetric sets, its sign depends on the parity of the permutation implicitly, so the variables in this element are called alternating. In the above notation we have the equality:  $\bar{x}_{i(1)} \dots \bar{x}_{i(k)} \bar{x}_{i(k+1)} \dots \bar{x}_{i(n)} = -\bar{x}_{i(1)} \dots \bar{x}_{i(k+1)} \bar{x}_{i(k)} \dots \bar{x}_{i(n)}$ , i.e. variables of alternating set can be interchanged by changing the sign of element on the opposite. According to Leibnitz identity can be written:  $\bar{x}_1 \bar{x}_2 \bar{x}_3 \bar{x}_4 \equiv \bar{x}_1 \bar{x}_2 \bar{x}_4 \bar{x}_3 + \bar{x}_1 \bar{x}_2 (\bar{x}_3 \bar{x}_4)$ . directly obtain:  $\bar{x}_1 \bar{x}_2 \bar{x}_3 \bar{x}_4 \equiv \frac{1}{2} \bar{x}_1 \bar{x}_2 (\bar{x}_3 \bar{x}_4)$ , and more generally:  $\bar{x}_1 \bar{x}_2 \dots \bar{x}_{2n+1} \equiv \frac{1}{2^n} \bar{x}_1 (\bar{x}_2 \bar{x}_3) \dots (\bar{x}_{2n} \bar{x}_{2n+1})$ . In other words, starting from the second place, variables of one skew-symmetric set, standing close by, we can combine in parentheses multiplying element on  $\frac{1}{2}$  for each pair.

In the case of zero characteristic of the base field every identity is equivalent to the system of multilinear identities, which is obtained using the standard method of linearization [4]. Here is an example of this process for the identity

$$x_0(xy)(xy) \equiv 0.$$

After linearization on variable  $x$ , we obtain:

$$x_0(x_1 y)(x_2 y) + x_0(x_2 y)(x_1 y) \equiv 0.$$

Full linearization is as follows:

$$x_0(x_1 y_1)(x_2 y_2) + x_0(x_1 y_2)(x_2 y_1) + x_0(x_2 y_1)(x_1 y_2) + x_0(x_2 y_2)(x_1 y_1) \equiv 0.$$

Arrange the linearization of the element  $f$  to designate  $lin f$ .

Recall that the Lie algebra is called engel if it satisfies  $xY^m \equiv 0$ . If the algebra  $A$  satisfies to identity  $x_1x_2 \dots x_{c+1} \equiv 0$ , but not satisfies  $x_1x_2 \dots x_c \equiv 0$ , it's called nilpotent of the class not more than  $c$ . Save these definitions in case of Leibnitz algebras.

## 2. EXAMPLE OF NOT NILPOTENT VARIETY OF LEIBNITZ ALGEBRAS WITH THE CONDITION OF ENGEL

In the case of zero characteristic of base field E.I. Zelmanov in paper [3] has proved, that an engel Lie algebra is nilpotent. Using this fact, in the paper [7] second author proved that in the case of zero characteristic of base field an engel Leibnitz algebra is nilpotent.

In the case of nonzero characteristic  $p$  of base field P.M. Cohn has given an example of a non-nilpotent Lie algebra over a residue field on a prime module  $\mathbb{Z}_p$ , which satisfies identities  $(xy)(zt) \equiv 0$  and,  $xY^{p+1} \equiv 0$ , that is an example of a non-nilpotent Engel metabelian Lie algebra.

Following the ideas of the article we construct a non-nilpotent metabelian Leibnitz algebra. Let  $\Phi$  be a field of prime characteristic  $p$ .

**Theorem.** A variety  $\mathbf{V}$  of Leibnitz algebras over field  $\Phi$ , satisfies to the identities  $xY^p \equiv 0$  and  $x(yz) \equiv 0$  is non-nilpotent.

Proof. We prove that over the field  $\Phi$  there is a non-nilpotent Leibnitz algebra  $M$ , which satisfies to the identities  $xY^p \equiv 0$  and  $x(yz) \equiv 0$ .

Let  $W$  be a vector space over a field  $\Phi$ , which has a basis  $\{e_f | f \in \Phi^{\mathbb{N}}\}$ , where  $\Phi^{\mathbb{N}}$  is the set of all functions of natural argument with values in  $\Phi$ . We define on the space  $W$  multiplication, assuming that the algebra  $W$  is an abelian Lie algebra, i.e.  $e_{f_1}e_{f_2} \equiv 0$ . For any natural number  $m$  we denote by  $\delta_m$  endomorphism of vector space  $W$ , which on the basic element  $e_f$  take a value  $e_{\bar{f}}$ , where

$$\bar{f}(i) = \begin{cases} f(i), & i \neq m, \\ f(i) + 1, & i = m. \end{cases}$$

It's easy to verify that  $\delta_m\delta_n = \delta_n\delta_m$   $\delta_m^p = \varepsilon$ ,  $\varepsilon$  is identical endomorphism of vector space  $W$ . Let  $x_i = \delta_i - \varepsilon$ ,  $i = 1, 2, \dots$ , than  $x_ix_j = (\delta_i - \varepsilon)(\delta_j - \varepsilon) = \delta_i\delta_j - \delta_i - \delta_j + \varepsilon = \delta_j\delta_i - \delta_j - \delta_i + \varepsilon = (\delta_j - \varepsilon)(\delta_i - \varepsilon) = x_jx_i$ .

Under the commutation operation  $L = \langle x_i | i \in \mathbb{N} \rangle_{\Phi}$  — the  $\Phi$ -hull of the set  $\{x_i | i \in \mathbb{N}\}$  is an abelian Lie algebra, and  $W$  is a right  $L$ -module.

Required Leibnitz algebra is a direct sum of vector spaces  $W$  and  $L$ , in which multiplication defined by the rule:

$$(w_1 + l_1)(w_2 + l_2) = w_1l_2,$$

where  $w_1, w_2 \in W$ ,  $l_1, l_2 \in L$ ,  $w_1l_2$  is a result of applying  $l_2$  to the element  $w_1$ , belonging to a vector space  $W$ . Denote the resulting algebra  $M$ .

The algebra  $M$  satisfies to the identity  $x(yz) \equiv 0$ . Indeed, substituting in a verifiable identity elements  $w_i + l_i \in M$ ,  $i = 1, 2, 3$ , we get

$$(w_1 + l_1)((w_2 + l_2)(w_3 + l_3)) = (w_1 + l_1)(w_2 l_3) = 0.$$

For any elements  $t, y$  of algebra  $M$  it is true the engel identity  $tY^p \equiv 0$ . Indeed, it's well known, that the binomial coefficient  $\binom{p}{m} = \frac{p!}{m!(p-m)!}$  is divided into  $p$ . Therefore for any  $i \in \mathbb{N}$ ,  $f \in K^{\mathbb{N}}$  by the binomial formula we obtain  $e_f X_i^p = e_f \underbrace{(\delta_i - \varepsilon)(\delta_i - \varepsilon) \dots (\delta_i - \varepsilon)}_p = e_f \underbrace{\delta_i \delta_i \dots \delta_i}_p - e_f \underbrace{\varepsilon \varepsilon \dots \varepsilon}_p = 0$ . The last equality follows from the fact that  $e_f \underbrace{\delta_i \delta_i \dots \delta_i}_p = e_f$ . Note that the result is

true in the case  $p = 2$ , as  $e_f x_i x_i = e_f \delta_i \delta_i + e_f \delta_i \delta_i = e_f \delta_i \delta_i - e_f \delta_i \delta_i = 0$ . Namely,  $e_f X_i^p \equiv 0$ , for any natural  $i$  and any function  $f$ . Let  $y = \sum_s \alpha_s x_s + \sum_f \beta_f e_f$  be an arbitrary element  $M$ , than  $tY^p = t(\sum_s \alpha_s X_s)^p = \sum_s \alpha_s^p tX_s^p = 0$ .

Now we verify that  $M$  is non-nilpotent Leibnitz algebra. Denote by  $f_{j_1, j_2, \dots, j_s}$ , where  $\{j_1, j_2, \dots, j_s\}$  is a strictly increasing set of natural numbers, the function of natural argument, which takes the value 1 at points  $j_1, j_2, \dots, j_s$  and value 0 in the rest of the points, and by  $f_0$  we denote the function, which takes the zero value in all points. We prove by induction on the number of factors the following formula

$$e_{f_0} x_1 x_2 \dots x_m = \sum_{k=0}^m (-1)^k \sum_{\{j_1, j_2, \dots, j_{m-k}\}} e_{f_0} \delta_{j_1} \delta_{j_2} \dots \delta_{j_{m-k}}, \quad (2)$$

where  $j_1 < j_2 < \dots < j_{m-k}$ ,  $\{j_1, j_2, \dots, j_{m-k}\} \subset \{1, 2, \dots, m\}$ . For  $m = 1$  we obtained  $e_{f_0} x_1 = e_{f_0} \delta_1 - e_{f_0} = e_{f_1} - e_{f_0}$  and formula (2) is true. Assume that the desired equality holds for  $m - 1$ , i.e.

$$e_{f_0} x_1 x_2 \dots x_{m-1} = \sum_{k=0}^{m-1} (-1)^k \sum_{\{j_1, j_2, \dots, j_{m-1-k}\}} e_{f_0} \delta_{j_1} \delta_{j_2} \dots \delta_{j_{m-1-k}}.$$

We prove that the equality (2) is true for  $m$ . Multiply both sides of last equation by  $x_m$ , we obtain

$$\begin{aligned} e_{f_0} x_1 x_2 \dots x_m &= \left( \sum_{k=0}^{m-1} (-1)^k \sum_{\{j_1, j_2, \dots, j_{m-1-k}\}} e_{f_0} \delta_{j_1} \delta_{j_2} \dots \delta_{j_{m-1-k}} \right) x_m = \\ &= \sum_{k=0}^m (-1)^k \sum_{\{j_1, j_2, \dots, j_{m-k}\}} e_{f_0} \delta_{j_1} \delta_{j_2} \dots \delta_{j_{m-k}} \end{aligned}$$

Thus, the formula (2) is true for any natural  $m$ .

From the equality  $e_{f_0} \delta_{j_1} \delta_{j_2} \dots \delta_{j_{m-k}} = e_{f_{j_1, j_2, \dots, j_{m-k}}}$  and the formula (2) we obtained that

$$e_{f_0} x_1 x_2 \dots x_m = \sum_{k=0}^m \sum_{\{j_1, j_2, \dots, j_{m-k}\}} (-1)^k e_{f_{j_1, j_2, \dots, j_{m-k}}}.$$

The last element is a linear combination of various basic elements with coefficients 1 or -1, therefore, it is different from zero for any natural  $m$ . The theorem is proved.

**Note.** If  $M$  is a Lie algebra, than the identity  $x(yz) \equiv 0$  is equal to the identity  $yzx \equiv 0$ , and the algebra  $M$  is nilpotent.

### 3. THE STRUCTURE OF MULTILINEAR PART OF VARIETY OF LEFT NILPOTENT OF THE CLASS NOT MORE THAN 3 LEIBNITZ ALGEBRAS

In this section we will consider the variety of Leibnitz algebras satisfying the identity

$$x(y(z t)) \equiv 0, \quad (3)$$

which we will denote by  ${}_3\mathbf{N}$ .

The variety  ${}_3\mathbf{N}$  has been studied by some authors (see, e.g., [1], [6], [8]). In particular, it was proved that the variety  ${}_3\mathbf{N}$  has only subvariety  $\tilde{V}_2$  and  $\tilde{V}_3$  with almost polynomial growth (see [6]); was to find a basis of multilinear part of variety  ${}_3\mathbf{N}$  (see [1]); were found multiplicity  $m_\lambda$  in the decomposition  $\chi({}_3\mathbf{N})$  (see [8]).

Here is an example of the Leibnitz algebra lying in the manifold  ${}_3\mathbf{N}$ . Let  $T_s = K[t_1, \dots, t_s]$  be a ring of polynomial from variable  $t_1, \dots, t_s$ . Consider the Heisenberg algebra  $H_s$  with basis  $\{a_1, \dots, a_s, b_1, \dots, b_s, c\}$  and multiplying  $a_i b_j = \delta_{ij} c$ , where  $\delta_{ij}$  is Kronecker delta, the product of other basic elements is equal to zero. It's well known (see, e.g. [1]), that the algebra  $H_s$  is nilpotent of the class 2 Lie algebra. Transform the polynomial ring  $T_s$  in the right module of algebra  $H_s$ , in which the basic elements of  $H_s$  act right on the polynomial  $f$  from  $T_s$  as follows:

$$f a_i = f'_i, f b_i = t_i f, f c = f,$$

where  $f'_i$  is a partial derivative of polynomial  $f$  in the variable  $t_i$ . It's easy to prove that the direct sum of vector space  $H_s$  and  $T_s$  with multiplication:

$$(x + f)(y + g) = xy + fy,$$

where  $x, y$  are from  $H_s$ ;  $f, g$  are from  $T_s$ , is a Leibnitz algebra. Denote this algebra by symbol  $H^s$ . The resulting algebra  $H^s$  belongs to the variety  ${}_3\mathbf{N}$  for any  $s$ . Prove that the identity (3) holds in  $H^s$ :

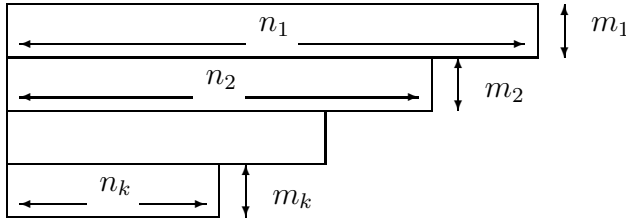
$$(x_1 + f_1)((x_2 + f_2)((x_3 + f_3)(x_4 + f_4))) = x_1(x_2(x_3 x_4) + f_1(x_2(x_3 x_4))) = 0.$$

This equality is true by the nilpotency of class 2 of algebra  $H_s$ .

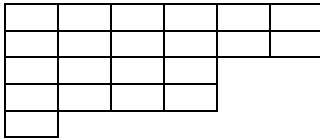
We define the general form of the elements that are not equal to zero. Since the Heisenberg algebra  $H_s$  is nilpotent of class 2, then the product of any tree elements is equal to zero. Therefore in algebra  $H^s$  all of elements of degree 3 and more must contain at least one polynomial from  $T_s$ . As  $T_s$  is a right module of algebra  $H_s$ , then the elements of form  $x_1 f$  are zero. Besides if the element of algebra  $H^s$  has two polynomials from  $T_s$ , then he also is zero according to the definition of the algebra  $H^s$ . Therefore all nonzero elements of algebra  $H^s$  of degree more than two must to have exactly one polynomial at the first place. If the element of algebra  $H^s$  has the polynomial outside of the first skew-symmetric set, then it can be written as the sum of terms, each of which not contains a polynomial in the first place and so it's equal to zero. Therefore nonzero elements of algebra  $H^s$  have the polynomial  $f$  in the first skew-symmetric set.

As previously mentioned, the space of multilinear elements of degree  $n$  of some variety of Leibnitz algebras is a direct sum of irreducible submodules, corresponding to all Young diagrams, which contain  $n$  cells; moreover, two module are isomorphic if and only if they meet the same diagram. In the paper [1] it was proved, that the number of isomorphic terms in specified sum for the space  $P_n(3\mathbf{N})$  is equal to the number of corner cell corresponding Young diagram. Therefore we will consider the diagrams with fixed number of corner cell.

Given a Young diagram with  $n$  cells, which has  $k$  corner cells, i.e. it responds to the partition  $\lambda = (n_1^{m_1}, n_2^{m_2}, \dots, n_k^{m_k})$ , where  $n_1 > n_2 > \dots > n_k > 0$  and  $n_1 m_1 + n_2 m_2 + \dots + n_k m_k = n$ , that is the diagram of form:



Consider the special case of diagrams of this form. Let  $n = 21$  and  $\lambda = (6, 6, 4, 4, 1)$ . Then responding diagram has form:



To this diagram respond next elements:

$$\begin{aligned}
 g_1 &= \bar{x}_1 \bar{x}_2 \bar{x}_3 \bar{x}_4 \bar{x}_5 \tilde{x}_1 \tilde{x}_2 \tilde{x}_3 \tilde{x}_4 \hat{x}_1 \hat{x}_2 \hat{x}_3 \hat{x}_4 \bar{\bar{x}}_1 \bar{\bar{x}}_2 \bar{\bar{x}}_3 \bar{\bar{x}}_4 \tilde{\tilde{x}}_1 \tilde{\tilde{x}}_2 \hat{\hat{x}}_1 \hat{\hat{x}}_2, \\
 g_2 &= \bar{x}_1 \bar{x}_2 \bar{x}_3 \bar{x}_4 \tilde{x}_1 \tilde{x}_2 \tilde{x}_3 \tilde{x}_4 \hat{x}_5 \hat{x}_1 \hat{x}_2 \hat{x}_3 \hat{x}_4 \bar{\bar{x}}_1 \bar{\bar{x}}_2 \bar{\bar{x}}_3 \bar{\bar{x}}_4 \tilde{\tilde{x}}_1 \tilde{\tilde{x}}_2 \hat{\hat{x}}_1 \hat{\hat{x}}_2, \\
 g_3 &= \bar{x}_1 \bar{x}_2 \tilde{x}_1 \tilde{x}_2 \tilde{x}_3 \tilde{x}_4 \hat{x}_5 \hat{x}_1 \hat{x}_2 \hat{x}_3 \hat{x}_4 \bar{\bar{x}}_1 \bar{\bar{x}}_2 \bar{\bar{x}}_3 \bar{\bar{x}}_4 \tilde{\tilde{x}}_1 \tilde{\tilde{x}}_2 \tilde{\tilde{x}}_3 \tilde{\tilde{x}}_4 \hat{\hat{x}}_1 \hat{\hat{x}}_2.
 \end{aligned}$$

Using the notation introduced earlier for the operators, agree standard polynomial of operators  $X_1, X_2, \dots, X_m$  denoted by  $\overline{St}_m = \overline{St}_m(X_1, \dots, X_m) = \overline{X}_{p(1)} \dots \overline{X}_{p(n)}$ . Note that the standard polynomials, containing different alternating set of variables, we will write with different upper symbols. In this designation it also true the generalization to the case of degree of standard polynomial from operators. Then the elements  $g_1, g_2$  and  $g_3$  we can write as follows:

$$\begin{aligned} g_1 &= \overline{x}_1 \overline{x}_2 \overline{x}_3 \overline{x}_4 \overline{x}_5 \widetilde{St}_4^3 \widehat{St}_2^2, \\ g_2 &= \overline{x}_1 \overline{x}_2 \overline{x}_3 \overline{x}_4 \widetilde{St}_5 \widehat{St}_4^2 \overline{\overline{St}}_2^2, \\ g_3 &= \overline{x}_1 \overline{x}_2 \widetilde{St}_5 \widehat{St}_4^3 \overline{\overline{St}}_2. \end{aligned}$$

We will return to the general case. Let as before the Young diagram respond to the partition  $\lambda = (n_1^{m_1}, n_2^{m_2}, \dots, n_k^{m_k})$ , where  $n_1 > n_2 > \dots > n_k > 0$  and  $n_1 m_1 + n_2 m_2 + \dots + n_k m_k = n$ . Then to it corresponds the elements of following form:

$$\begin{aligned} g_1 &= (\overline{x}_1 \overline{x}_2 \dots \overline{x}_{d_k}) \widetilde{St}_{d_k}^{n_k-1} \widehat{St}_{d_{k-1}}^{n_{k-1}-n_k} \dots \overline{\overline{St}}_{d_2}^{n_3-n_2} \widetilde{St}_{d_1}^{n_2-n_1}, \\ g_2 &= (\overline{x}_1 \overline{x}_2 \dots \overline{x}_{d_{k-1}}) \widetilde{St}_{d_k}^{n_k} \widehat{St}_{d_{k-1}}^{n_{k-1}-n_k-1} \dots \overline{\overline{St}}_{d_2}^{n_3-n_2} \widetilde{St}_{d_1}^{n_2-n_1}, \\ &\dots \\ g_k &= (\overline{x}_1 \overline{x}_2 \dots \overline{x}_{d_1}) \widetilde{St}_{d_k}^{n_k} \widehat{St}_{d_{k-1}}^{n_{k-1}-n_k} \dots \overline{\overline{St}}_{d_2}^{n_3-n_2} \widetilde{St}_{d_1}^{n_2-n_1-1}, \end{aligned}$$

where  $d_j = \sum_{i=1}^j m_i$ ,  $j = 1, \dots, k$ .

It is known (see.[9], chapter 2.4, p.54), that the linearization of any element  $g_m(\lambda)$ , where  $m = 1, \dots, k$ , generates an irreducible module  $W_m(\lambda) = KS_n(\text{ling}_m(\lambda))$ , corresponding to the fixed partition  $\lambda$ .

The main result of this section is the prove of the fact, that the elements  $g_1, g_2, \dots, g_k$  generate the irreducible  $S_n$ -modules that provide the direct sum for the multilinear part  $P_n$ .

**Theorem.** For any natural number  $n \geq 1$  it's true the equality

$$P_n = \bigoplus_{\lambda \vdash n} \bigoplus_{r=1}^{k(\lambda)} W_r(\lambda). \quad (4)$$

Proof. In the paper [5] it was proved that the number of isomorphic irreducible submodules in the formula (4) is equal to the maximum number of linearly independent elements of form  $g_i$ ,  $i = 1, \dots, k$ , which generate these submodules. So for the proof of theorem it suffices to prove the linear independence of elements  $g_1, \dots, g_k$ . Assume the contrary. Suppose that there is a linear relationship

$$\alpha_1 g_1 + \alpha_2 g_2 + \dots + \alpha_k g_k = 0, \quad (5)$$

where At least one of  $\alpha_i$ ,  $i = 1, \dots, k$  is non-zero. Suppose that  $l$  is the smallest number such that  $\alpha_l \neq 0$  and  $\alpha_q = 0$ , if  $q < l$ . Now let  $d_j = 2p_j + \varepsilon_j$

where  $\varepsilon_j = 1$ , if  $d_j$  is odd and  $\varepsilon_j = 0$ , if  $d_j$  is even. Will carry out the substitution variables in place of elements  $g_l, \dots, g_k$  of elements from algebra  $H^s$  as follows: if  $\varepsilon_j = 1$ , that  $x_{d_j} = a_{p_j+1}$  ( $j \neq k$ ),  $x_{d_k} = f + a_{p_k+1}$ ; if  $\varepsilon_j = 0$ , that  $x_{d_j} = b_{p_j}$  ( $j \neq k$ ),  $x_{d_k} = f + b_{p_k}$ . Elements resulting from the substitution from  $g_l, \dots, g_k$ , we will denote by  $v_l, \dots, v_k$  respectively. Then the elements  $v_{l+1}, \dots, v_k$  are zero by definition of algebra  $H^s$ , as well as of their structure can be seen, that they do not contain a polynomial  $f$  in the first place.

We will consider now following private type of an element

$$g_l = \overline{x_1} \overline{x_2} \overline{x_3} \overline{x_4} \overline{x_5} \overline{x_6} \overline{\widetilde{x_1}} \overline{\widetilde{x_2}} \overline{\widetilde{x_3}} \overline{\widetilde{x_4}} \overline{\widetilde{x_5}} \overline{\widetilde{x_6}} \widetilde{x_1} \widetilde{x_2} \widetilde{x_3} \widetilde{x_4} \widetilde{\widetilde{x_1}} \widetilde{\widetilde{x_2}} \widetilde{\widetilde{x_3}}.$$

After substituting its variables, we obtain an element:

$$\begin{aligned}
v_l &= \bar{a}_1 \bar{b}_1 \bar{a}_2 \bar{b}_2 \bar{a}_3 f + b_3 \bar{a}_1 \bar{b}_1 \bar{a}_2 \bar{b}_2 \bar{a}_3 \overline{f} + \overline{b_3 \bar{a}_1 \bar{b}_1 \bar{a}_2 \bar{b}_2 \bar{a}_3 f} + \overline{b_3 \bar{a}_1 \bar{b}_1 \bar{a}_2 \bar{b}_2 \bar{a}_3 \overline{f}} \\
&\equiv (-1)^5 f \bar{a}_1 \bar{b}_1 \bar{a}_2 \bar{b}_2 \bar{a}_3 \bar{a}_1 \bar{b}_1 \bar{a}_2 \bar{b}_2 \bar{a}_3 \bar{b}_3 \tilde{a}_1 \tilde{b}_1 \tilde{a}_2 \tilde{b}_2 \tilde{a}_1 \tilde{b}_1 \tilde{a}_2 \\
&\equiv \frac{-1}{2^8} f (\bar{a}_1 \bar{b}_1) (\bar{a}_2 \bar{b}_2) \bar{a}_3 (\bar{a}_1 \bar{b}_1) (\bar{a}_2 \bar{b}_2) (\bar{a}_3 \bar{b}_3) (\tilde{a}_1 \tilde{b}_1) (\tilde{a}_2 \tilde{b}_2) (\tilde{a}_1 \tilde{b}_1) \tilde{a}_2 \\
&\equiv \frac{-1}{2^8} f ccccccca_3 a_2 \equiv \frac{-1}{2^8} f a_3 a_2.
\end{aligned}$$

We return to the general case. According to the arguments given earlier, in the element  $v_l$  are non-zero only those terms, that contain exactly one polynomial  $f$  in the first place; that is among the first  $n_k$  of a skew-symmetric sets (which include one of the sums  $f + a_{l_k+1}$  or  $f + b_{p_k}$ ), only the first set contains a polynomial  $f$ , and the rest contains the second term  $a_{p_k+1}$  or  $b_{p_k}$ . In the first skew-symmetric set the polynomial  $f$  is initially last. So that it was on the first place, it's necessary  $d_k - 1$  times rearrange it with elements to the left of it. So the element  $v_l$  will receive a coefficient  $(-1)^{d_k-1}$ . In addition the element  $v_l$  after polynomial  $f$  or elements  $a_i b_i$ , standing near, or elements  $a_i$  that stand alone. The pair of elements  $a_i b_i$  combine in brackets, so that each pair gives the coefficient  $\frac{1}{2}$ . The product of elements  $a_i b_i$  inside the brackets is equal to  $c$ , since the Kronecker delta  $\delta_{ii}$  is equal to 1. Since the multiplication polynomial  $f$  by  $c$  gives again a polynomial  $f$ , then in element  $v_l$  to the right of  $f$  will remain only elements  $a_i$  that stand outside the brackets. Therefore, the element  $v_l$  will have the form :

$$v_l = \frac{(-1)^{d_k-1}}{2^r} f a_{p_k+\varepsilon_k}^{\varepsilon_k(n_k-2)+1} a_{p_{k-1}+1}^{\varepsilon_{k-1}(n_{k-1}-n_k)} \dots a_{p_1+1}^{\varepsilon_1(n_1-n_2)},$$

where  $r = p_1(n_1 - n_2) + p_2(n_2 - n_3) + \cdots + p_{k-1}(n_{k-1} - n_k) + p_k n_k + \varepsilon_k - 1$ . The polynomial  $f$  we can select so that the result of its differentiation on variables  $t_1, \dots, t_s$  was nonzero. Thus, the element  $v_l$  is nonzero. Note that the elements  $v_{l+1}, \dots, v_k$  in such a substitution are equal to zero, because they does not contain a polynomial of the rings  $T_s$  in the first skew-symmetric set.

Let us return to the linear combination (5). By assumption  $\alpha_1 = \alpha_2 = \dots = \alpha_{l-1} = 0$ . After making this substitution this linear combination will have the



form:  $\alpha_l \cdot v_l + \alpha_{l+1} \cdot 0 + \dots + \alpha_k \cdot 0 = 0$ , where  $v_l \neq 0$ . Therefore,  $\alpha_l = 0$ . We received a contradiction with the assumption. Therefore, all the coefficients  $\alpha_1, \alpha_2, \dots, \alpha_k$  in the linear combinations (5) are equal to zero, that is, elements  $g_1, g_2, \dots, g_k$  are linearly independent. The theorem is proved.

The authors would like to thank their supervisor S.P. Mishchenko for the formulation of a problem, useful tips and attention to the paper.

## REFERENCES

- [1] Abanina L., Structure and identities of some varieties of Leibnitz algebras [Text]: Dissertation for the degree of candidate of physic- mathematical sciences: 01.01.06: defended 16.06.03: approved 28.10.03 / Abanina Lyubov Evgenyevna. Ulyanovsk, UISU, 2003. 65 p. Bibliography: p. 61–65.
- [2] Blokh ..., A generalization of the concept of a Lie algebra, Dokl. akad. nauk. SSSR, 18(1965), no. 3, 471–473.
- [3] Zel'manov .I., About engel Lie algebras, Reports of the USSR Academy of Sciences, 292(1987), no 2, 265–268.
- [4] Mal'tsev A.I., On algebras defined by identities, Mat. Sat, 26(1950), no. 1, 19–33.
- [5] Mishchenko S.P., Zaicev M.V., On the growth of the identities of algebras, Algebra Discrete Math, 79(2006), no. 4, 553–559.
- [6] S.P. Mishchenko, T.V. Shishkina, The Leibnitz algebras of almost polynomial growth with the identity  $x(y(zt)) \equiv 0$ , Vestnik Moskovskogo Universiteta. Seriya 1. Matematika. Mekhanika, 2010, no. 3, 18–23; English translation: Moscow Univ. Math. Bulletin, 65(2011), no. 3, 107–110.
- [7] Frolova Yu.Yu., On nilpotentness of Leibnitz algebras with Engel condition, Vestnik Moskovskogo Universiteta. Seriya 1. Matematika. Mekhanika, 2011, no. 3, 63–65.
- [8] Abanina L.E., Mishchenko S.P., The variety of Leibnitz algebras defined by the identity  $x(y(zt)) = 0$ , Serdika Math. J., 9(2003), no. 29, 291–300.
- [9] Giambruno A., Zaicev M.V., Polynomial Identities and Asymptotic Methods, Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, 122(2005).
- [10] Cohn P.M., A non-nilpotent Lie ring satisfying the Engel condition and a non-nilpotent Engel group, Proc. Cambridge Phil. Soc.: Math. and Phys.Sci., 51(1955), no. 3, 401–405.